The fractional Ornstein–Uhlenbeck process as a representation of homogeneous Eulerian velocity turbulence

Yaping Shao

Centre for Advanced Numerical Computation in Engineering and Science, University of New South Wales, Sydney 2052, NSW, Australia

Received 30 November 1993; revised 23 January 1995; accepted 23 January 1995

Communicated by U. Frisch

Abstract

A fractional Langevin equation is an analogy to the Langevin equation but with fractional Gaussian noise as the source of randomness. The fractional Ornstein–Uhlenbeck process determined by the fractional Langevin equation is a stationary Gaussian process with a structure function which may differ from being proportional to time increment depending on a characteristic model parameter $H$. Such a model can be applied to random processes in turbulent flows by varying $H$, including homogeneous Eulerian and Lagrangian turbulence ($H = 1/3$ and $1/2$, respectively). Theoretical analysis, numerical tests and comparisons between simulation and observation show that with $H = 1/3$, the fractional Ornstein–Uhlenbeck process reproduces the basic features of homogeneous Eulerian turbulence. The model provides a promising technique for describing the diffusion of nonpassive particles in turbulent flows.

1. Introduction

In Lagrangian stochastic models for turbulent diffusion of heavy particles, it is essential to describe adequately the trajectory-crossing effect which arises from the fact that heavy particles, owing to their finite mean velocity, continuously change their fluid-particle neighbourhood [1, 2]. This problem involves the estimation of fluid velocity fluctuations along the heavy particle trajectories and thus, as discussed in Shao and Raupach [3], a requirement for stochastic simulation of turbulence with Eulerian statistical properties. This study explores the possibility of using a fractional stochastic differential equation as a representation of homogeneous Eulerian turbulence. This approach is an extension of recent studies on fractional Brownian motions [4, 5] which have made a considerable impact on the development of fluid mechanics and the interpretation of turbulence, especially in quantifying the geometric properties of the turbulence field [6, 7].

Most Lagrangian stochastic models for turbulent diffusion of scalars are based on the Ito stochastic differential equation of the form

$$\mathrm{d} \chi = a(\chi, t) \, \mathrm{d}t + b(\chi, t) \, \mathrm{d}\Omega,$$

where $\chi$ is a random process of time, $t$, $a(\chi, t)$ is the drift coefficient characterising the deterministic acceleration, and $b(\chi, t)$ is the diffusion coefficient characterising the random acceleration.
tion. The Wiener process (or Brownian motion), \( \mathbf{W} \), has uncorrelated increments \( d\mathbf{W} \), which obey

\[
\langle d\mathbf{W}(t) \cdot d\mathbf{W}(t+\tau) \rangle = \delta(\tau) dt.
\]

where the operator \( \langle \cdot \rangle \) represents an ensemble average and \( \delta(\tau) \) is either 1 (for \( \tau = 0 \)) or 0 (for \( \tau \neq 0 \)). The mathematical model for \( d\mathbf{W} \) is Gaussian white noise with a \( \delta \)-auto-correlation function. The uncorrelated nature of the increments makes Brownian motion the most fundamental stochastic process in the classical theory of stochastic processes.

One of the simplest forms of Eq. (1) is the one-dimensional Langevin equation with a linear drift coefficient \( -kX \) and a constant diffusion coefficient \( D \)

\[
dX = -kX \, dt + \sqrt{D} \, d\mathbf{W}.
\]

The OU (Ornstein-Uhlenbeck) process described by Eq. (3) is a Gaussian process with zero mean and variance \( D/2k \). This process has an exponential auto-correlation function and a structure function proportional to time increment. Eq. (3) is an adequate model for scalar diffusion in homogeneous turbulence [8]. More recently, stochastic models with a nonlinear drift coefficient have been developed by Thomson [9] and shown to be capable of modelling turbulent diffusion in convective flows.

While stochastic models based on (1) reproduce the most important statistical features of Lagrangian turbulence, they are not suitable for modelling the velocity fluctuations observed in the Eulerian system. This is because the exponential auto-correlation function and proportionality between the structure function and \( dt \) are the basic features of all Itô-type stochastic models, while the Eulerian structure function is not proportional to \( dt \). and the Eulerian auto-correlation function is not simply an exponential function.

Studies on fractional Brownian motions [4, 5] constructed a first step toward a stochastic simulation of random processes with peculiar small-time behaviour, such as the velocity fluctuation observed in the Eulerian framework. Mandelbrot [5] initiated an investigation on the geometry of the Eulerian turbulence field by using the concept of fractional Brownian motion and showed the usefulness of this approach. However, it is inadequate to compare directly a fractional Brownian motion with Eulerian turbulence as in [5], because the former is a non-stationary process with the variance increasing to infinity as \( t \to \infty \) while the latter must be considered as a stationary process with a finite variance.

In this study, a generalised Langevin equation in the following form is proposed for stochastic simulation of homogeneous turbulence:

\[
dX_b = -k_bX_b \, dt + \sqrt{D_b} \, d\mathbf{W}_b,
\]

where \( d\mathbf{W}_b \) is the increment of a fractional Brownian motion characterised by the Hurst parameter \( H \) (\( 0 < H < 1 \)). Eq. (4) is an extension of Eq. (3) with \( d\mathbf{W} \) being substituted by \( d\mathbf{W}_b \) and the two equations are identical for \( H = 1/2 \). Eq. (4) is referred to as the fractional Langevin equation and the stochastic process it represents as the fractional OU process. The generalised Langevin equation represents a possible step towards modelling turbulence by using the concept of fractional Brownian motions. The stochastic processes described by Eq. (4), in which a linear drift term is combined with a fractional Brownian motion, are now stationary processes with a finite variance. By varying \( H \), the model can be applied to a range of random physical processes, including turbulent diffusion of scalars (Lagrangian turbulence), but more importantly, it is capable of modelling stochastic processes with peculiar small-time properties such as Eulerian turbulence, which was the original motivation of this paper.

2. Fractional Brownian motion and fractional Langevin equation

2.1. Fractional Brownian motion

It is appropriate first to outline the basic
properties of fractional Brownian motions relevant to this study. More detailed discussions can be found in [4, 10]. According to Mandelbrot and Van Nees [4], a fractional Brownian motion is defined as a moving average of the Wiener process,

\[ \dot{\Omega}_b(t) - \dot{\Omega}_b(0) = \frac{1}{\Gamma(H + 1/2)} \left( \int_{-\infty}^{t} (t-s)^{H-1/2} d\Omega(s) \right) - \int_{-\infty}^{0} (-s)^{H-1/2} d\Omega(s) \].

(5)

where \( \Gamma \) is a gamma function. \( H \) is the key parameter characterising the fractional processes and can be classified into three categories with distinct mathematical properties: for \( H < 1/2 \), \( H = 1/2 \) and \( H > 1/2 \). The increment of a fractional Brownian motion \( \dot{\Omega}_b(t + \tau) - \dot{\Omega}_b(t) \) is a stationary and isotropic Gaussian process with variance given by

\[ \langle |\dot{\Omega}_b(t + \tau) - \dot{\Omega}_b(t)|^2 \rangle = \nu \tau^{2H} . \]

(6)

where \( \nu \) is the key parameter. Fractional Brownian motions have statistically self-similar increments with parameter \( H \). This implies that if time \( t \) is changed by a factor of \( h \) then the increments of fractional Brownian motion change by a factor of \( h^H \); in other words \( \dot{\Omega}_b(t_0 + h\tau) - \dot{\Omega}_b(t_0) \) and \( h^{-H} [\dot{\Omega}_b(t_0 + h\tau) - \dot{\Omega}_b(t_0)] \) are statistically indistinguishable. In particular, the variance of the increments obeys

\[ \langle |\dot{\Omega}_b(t_0 + \tau) - \dot{\Omega}_b(t_0)|^2 \rangle = h^{-2H} \langle |\dot{\Omega}_b(t_0 + h\tau) - \dot{\Omega}_b(t_0)|^2 \rangle . \]

(7)

As for the Wiener process, a fractional Brownian motion \( \dot{\Omega}_b \) has continuous and nondifferentiable sample paths and may result from integration of fractional Gaussian noise (\( \omega_b \)) which can be formally considered as the derivative of \( \dot{\Omega}_b \), namely \( d\dot{\Omega}_b = \omega_b \, dt \). In general, the auto-correlation function of the increments of fractional Brownian motions is given by

\[ \langle \dot{\Omega}_b(t) \, d\dot{\Omega}_b(t + \tau) \rangle = \langle \omega_b(t) \, \omega_b(t + \tau) \rangle \, dt^2 \]

\[ = \frac{1}{2} \lambda^{2H-2} \tau^{2H} \left( \frac{\tau}{\lambda} + 1 \right)^{2H-2} \left| \frac{\tau}{\lambda} - 1 \right|^{2H-2} \]

(8)

where \( \lambda \) is a smoothing factor [4]. The energy spectrum of fractional Gaussian noise in general obeys

\[ F_{\omega_b}(f) \propto f^{-(2H+1)} . \]

(9)

where \( f \) is frequency. For the Wiener process, \( F_{\omega_b}(f) \) is a constant corresponding to a \( \delta \)-auto-correlation function. Since \( \dot{\Omega}_b \) can be considered as an integration of \( \omega_b \) the energy spectrum of a fractional Brownian motion is given by

\[ F_{\dot{\Omega}_b}(f) \propto f^{-(2H+1)} . \]

(10)

2.2. Fractional Langevin equation

The need for using fractional Gaussian noise instead of Gaussian white noise in Eq. (4) can be understood by comparing the statistical behaviour of Eulerian and Lagrangian turbulence. In the Eulerian system, the flow is described by velocity \( u_b(x, t) \) measured at location \( x \) at time \( t \) while in the Lagrangian system, it is described by the position of all individual fluid elements \( X(x_0, t) \) under the initial condition \( X(x_0, 0) = x_0 \). The Lagrangian velocity, \( v_b(x_0, t) = \frac{dX(x_0, t)}{dt} \), is related to \( u_b(x, t) \) by \( v_b(x_0, t) = u_b(X(x_0, t), t) \). Although it may seem intuitively plausible that Lagrangian and Eulerian turbulence are simply related, it is difficult in general to derive the Eulerian statistics from the Lagrangian statistics or vice versa [11]. Nevertheless, for homogeneous turbulence in an incompressible fluid, the Lagrangian and Eulerian probability density functions, \( p(v_b(x_0, t)) \) and \( p(u_b(x, t)) \), are identical [12] and thus the statistical moments of Lagrangian and Eulerian turbulence are identical.
\[ \int u_i v_i p(\nabla, (x_i, t)) \, dv_i = \int u_i v_i p(u_i, t) \, du_i. \]

The statistical difference between Lagrangian and Eulerian turbulence can first be seen in their small-time behaviour [13]. According to Kolmogorov's theory of local isotropy, in the inertial subrange the Lagrangian velocity structure function obeys

\[ D_n(\tau) = \delta_n C_o \varepsilon \tau. \tag{11} \]

where \( \varepsilon \) is the mean energy dissipation rate and \( C_n \) is a universal constant. The Eulerian structure function defined by

\[ E_n(r) = \langle [u_i(x + r) - u_i(x)][u_i(x + r) - u_i(x)] \rangle \]

can be expressed in two scalar functions

\[ E_n(r) = \frac{E_n(r) - E_n(\tau)}{r^2} - \frac{r \varepsilon}{r} E_n(\tau) \delta_n. \]

where \( r = |r| \), while \( E_u \) and \( E_n \) are the structure functions of \( u_i \) (parallel \( u \) component to \( r \)) and \( u_n \) (perpendicular \( u \) component to \( r \)), respectively. We consider now the special case of \( r = \bar{u} \tau \) with \( \bar{u} \) being the mean flow velocity. If Taylor's hypothesis of frozen turbulence applies, then \( E_u \) and \( E_n \) obey the relationships

\[ E_u(\bar{u} \tau) = C_1 (\bar{u} \varepsilon \tau)^3. \]
\[ E_n(\bar{u} \tau) = \frac{3}{2} C_1 (\bar{u} \varepsilon \tau)^3. \tag{12} \]

where \( C_1 \) is another universal constant.

The equivalent formulation of Eqs. (11) and (12) in the spectral domain is that in the inertial subrange the Lagrangian and Eulerian energy spectra behave respectively as

\[ F_L(f) = C_0 f^{-2}, \quad F_I(f) = C_1 f^{-3} \tau^{-3}. \tag{13} \]

A related difference lies in the Lagrangian and Eulerian auto-correlation functions: while the former can be approximated as an exponential function,

\[ R_L(\tau) = \exp(-\tau/T_L), \tag{14} \]

where \( T_L \) is the Lagrangian time scale, the latter cannot be expressed by simple analytic functions. One theory for the Eulerian auto-correlation functions is to assume that both Eulerian and Lagrangian auto-correlation functions have the same exponential shape [14],

\[ R_I(\tau) = R_L(\beta \tau). \tag{15} \]

This assumption implies, however, that the spectral density for Lagrangian and Eulerian turbulence has the following relationship:

\[ FF_I(f) = \beta FF_L(\beta f). \tag{16} \]

which is inconsistent with (13). Despite this inconsistency, Eq. (17) finds its applications in some diffusion models for heavy particles [2, 15, 16].

The above discussion reveals that an Ito stochastic model is not a suitable representation of Eulerian turbulence, because the structure function of the stochastic process given by Eq. (1) is determined by the Wiener increments which lead to a structure function proportional to \( \tau \) (Eq. (11)), rather than to \( \tau^{2.5} \) (Eq. (12)). Further, it is not possible to obtain from Eq. (3) an auto-correlation function which differs from the exponential form.

In contrast, the increment of a fractional Gaussian noise provides a promising representation for the increments of Eulerian velocity fluctuations. A comparison between Eqs. (6) and (12) suggests that the inertial subrange behaviour of Eulerian turbulence can be described by the increments of the fractional Brownian motion with \( H = 1/3 \). Such an analogy has been used in a recent study [7] of the fractional structure of turbulence. The geometric argument for using fractional Gaussian noise in (4) is the fractional nature of turbulence: the iso-surfaces of a turbulence field are folded and fragmented in such a complex way that they can be best considered as having fractional dimensions; and
the underlying physical argument is the self-
similarity property of small-scale turbulence re-
lected in the inertial subrange structure func-
tions. The self-similarity of small-scale turbu-
lence occurs because the cascade breaking down
of turbulence is the dominant process in the
inertial subrange. Assuming that Eq. (12) is
correct (we shall not consider here the refined
hypotheses on Eulerian structure functions which
include turbulence intermittency, e.g. Borgas
[17]), then we have \( E(r) = C_1(r)^{2/3} \) and \( E(l) =
C_1(l)^{2/3} \). For both \( r \) and \( l \) in the inertial sub-
range and \( l = hr \), it follows that
\[
E(r) = h^{-2/3} E(l) ,
\]
which is precisely the self-similar property de-
scribed by Eq. (7) for \( H = 1/3 \).

However, a direct comparison between frac-
tional Brownian motion and Eulerian turbulence is
inadequate since fractional Brownian motions are
nonstationary random processes with infinite
variance as \( t \to \infty \), while the variance of Eulerian
turbulence must be considered as finite. The
probability density function of fractional Brown-
ian motions obeys
\[
P(\Omega_h, t | \Omega_n, t_0) = [2\pi(t - t_0)^{2H}]^{-1/2} \times \exp \left( \frac{(\Omega_h - \Omega_n)^2}{2(t - t_0)^{2H}} \right)
\]
and thus the variance of the process increases
like \((t - t_0)^{2H}\). Neither is it plausible from the
viewpoint of physics to assume that the self-
similarity of small-scale turbulence can be ap-
plied to all scales. For turbulence with a length
scale smaller than Kolmogorov's length scale,
the kinetic energy is no longer determined by the
cascade process of turbulence as viscosity be-
comes progressively dominant for finer scales.
On large scales beyond the inertial subrange,
other external forces are important. Thus, turbu-
lence on large and small scales is not similar to
turbulence in the inertial subrange.

Therefore, it is necessary that the stochastic
model be formulated using fractional Gaussian
noise as the source of randomness, but also to
ensure a stationary process with finite variance.
It is interesting to notice that the Wiener process
is itself a nonstationary process with infinite
variance as \( t \to \infty \), while the OU process obtained
from a combination of a linear drift term to-
gether with the Wiener increments is a stationary
process with limited energy. It is thus natural to
suggest that a similar linear drift term might
ensure a stationary fractional process. It was
thought initially that a somewhat more com-
licated form than Eq. (4) would be required to
ensure stationarity and other suitable statistical
features (probability density function and corre-
lation characteristics). However, as will be
shown in the following sections, Eq. (4) turns
out to be sufficient.

3. The fractional Ornstein–Uhlenbeck process

3.1. Theoretical analysis

Several aspects of the fractional OU process
can be studied by theoretical analysis. If there is
no deterministic term, Eq. (4) is a fractional
Brownian motion, and if there is no random
acceleration, it represents a decaying process
with a relaxation time \( 1/k_h \).

\[
X_h(t) = x_h(t_0) e^{-k_h(t - t_0)} .
\]

In general, by substituting \( \xi_h = X_h e^{k_h(t - t_0)} \), the
solution of Eq. (4) is found to be formally
\[
X_h(t) = x_h(t_0) e^{-k_h(t - t_0)}
+ \sqrt{D_h} \int_{t_0}^{t} e^{-k_h(t' - t_0)} d\Omega_h(t') .
\]

The stochastic integral on the r.h.s. of Eq. (20)
requires some attention. To avoid the difficulty
of developing a new type of stochastic calculus,
we assume that this stochastic integral exists and
can be evaluated following the rules of Ito
One advantage of doing so is that for $H = 1/2$, Eq. (20) is the familiar OU process. Eq. (20) indicates that the fractional OU process is a Gaussian process with the mean given by

$$\langle x_b(t) \rangle = \langle x_b(t_0) \rangle e^{-k_b(t-t_0)}.$$  

(21)

which approaches zero for large $t-t_0$. Under the assumption that $x_b(t_0)$ is not correlated with $d\Omega_b$, the variance of the process is given by

$$\sigma_x^2 = \langle x_b(t)^2 \rangle = \langle x_b(t_0)^2 \rangle e^{-2k_b(t-t_0)}$$

$$+ D_b \left( \int_{t_0}^t e^{-k_b(t'-t)} d\Omega_b(t') \right)^2.$$  

(22)

and for sufficiently large $t-t_0$, it is determined only by the second term. Because the fractional Gaussian increments are correlated, it is difficult in general to provide an analytical expression for $\sigma_x^2$ except for $H = 1/2$, but some progress can be made by considering the correlation properties between the increments. For this purpose, we divide $t-t_0$ into $N$ equal intervals with an increment $\delta t$ and approximate the stochastic integral in (22) as a sum of finite discrete terms following the rules of Ito calculus. For simplicity, we also assume that $\langle x_b(t_0)^2 \rangle = 0$ so that the first term in (22) vanishes. Then

$$\sigma_x^2 = D_b \left( \sum_{j=1}^{N} \exp[-k_b(t-t_{j-1})] d\Omega_b(t_{j-1}) \right)$$

$$\times \left( \sum_{j=1}^{N} \exp[-k_b(t-t_{j-1})] d\Omega_b(t_{j-1}) \right).$$  

(23)

It follows that

$$\sigma_x^2 = D_b \left( \sum_{j=1}^{N} \exp[-2k_b(t-t_{j-1})] \delta \Omega_b(t_{j-1}) \right)$$

$$+ 2 \sum_{j=1}^{N} \sum_{i=1}^{j-1} \exp[-k_b(2t-t_{i-1}-t_{j-1})]$$

$$\times \delta \Omega_b(t_{i-1}) \delta \Omega_b(t_{j-1}) \right).$$  

(24)

In practice, a discrete fractional Gaussian noise is necessary for the numerical integration of Eq. (4). The discrete noise can be understood as a smoothed continuous noise with the smoothing factor $\lambda = \delta t$. As a result, Eq. (8) becomes

$$\langle d\Omega_b(t) d\Omega_b(t+\tau) \rangle$$

$$= \frac{1}{2} \delta t^{2H} \left( \left| \frac{\tau}{\delta t} + 1 \right|^{2H} + \left| \frac{\tau}{\delta t} - 1 \right|^{2H} - 2 \left| \frac{\tau}{\delta t} \right|^{2H} \right).$$  

(25)

It is finally obtained that

$$\sigma_x^2 = D_b \delta t^{2H} \left( \sum_{j=1}^{N} \exp[-2k_b(t-t_{j-1})] \right)$$

$$+ \sum_{j=1}^{N} \sum_{i=1}^{j-1} \exp[-k_b(2t-t_{i-1}-t_{j-1})]$$

$$\times \left( (j-i+1)^{2H} + (j-i-1)^{2H} - 2(j-i)^{2H} \right).$$  

(26)

The first sum in (26) is of a geometrical series and can be expressed as

$$D_b \delta t^{2H} \frac{1 - e^{-2k_b\delta t}}{e^{2k_b\delta t} - 1}.$$  

In the limit $\delta t \to 0$ this term has the following properties:

$$\lim_{\delta t \to 0} D_b \delta t^{2H} \frac{1 - e^{-2k_b\delta t}}{e^{2k_b\delta t} - 1} = \begin{cases} D_b/2k_b & H = 1/2, \\ 0 & H > 1/2 \end{cases}.$$  

Nevertheless, Eq. (26) can be readily evaluated numerically, and numerical tests show that for sufficiently large $t$, $\sigma_x^2$ approaches a constant; and for sufficiently small $\delta t$, the choice of $\delta t$ has little influence on the accuracy of $\sigma_x^2$ (see Fig. (1). Figs. 1a and 1b show the results of $\sigma_x^2$ for different choices of $\delta t$ with $H = 1/3$ and $2/3$, respectively. Term 1 and Term 2 are used to represent the first and the second sum in Eq. (26), and the parameters $k_b$ and $D_b$ are chosen arbitrarily. For $H = 1/3$, Term 1 increases and Term 2 decreases as $\delta t$ decreases, but $\sigma_x^2$ remains unchanged. For $H = 2/3$, Term 1 decreases and Term 2 increases as $\delta t$ decreases, and $\sigma_x^2$ again remains unchanged. The numerical tests suggest that both terms are well-behaved functions of $\delta t$, but the author has not been able...

Fig. 1. Numerical evaluation of $\sigma_{\epsilon}^2$ using Eq. (20). (a) Dependency of $\sigma_{\epsilon}^2$ on $\delta \tau$. The parameters $H = 1/3$, $D_\epsilon = 0.1$, $k_\epsilon = 0.1$ and $t = 6.25$ are arbitrarily chosen. (b) Same as (a) but with $H = 2/3$. (c) Dependency of $\sigma_{\epsilon}^2$ on $t$ for $H = 1/3$ and $2/3$ with $\delta \tau = 0.05$.

to derive simpler expressions for the second term. Fig. 1c shows that $\sigma_{\epsilon}^2$ is independent of $t$ for sufficiently large $i$.

Similarly, the auto-covariance function is given by

$$c(t, s) = \langle x_b(t) x_b(s) \rangle = D_b \left\langle \left( \int_{t_0}^t e^{-k_b(t-t')} d\Omega_b(t') \right) \left( \int_{t_0}^s e^{-k_b(s-t')} d\Omega_b(t') \right) \right\rangle. \tag{27}$$

Assuming $s > t$ and $s - t_0 = M \delta \tau$, Eq. (27) can be approximated by

$$c(t, s) = D_b \delta \tau^H \left( \sum_{i=1}^N \exp[-k_b(t + s - 2t_{i-1})] \right) + \sum_{i=1}^N \sum_{j=1}^\infty \exp[-k_b(t + s - t_{i-1} - t_{j-1})] \times \left[ (j - i + 1)^{2H} + (j - i - 1)^{2H} - 2(j - i)^{2H} \right]$$

$$+ \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^M \exp[-k_b(t + s - t_{i-1} - t_{j-1})] \times \left[ (j - i + 1)^{2H} + (j - i - 1)^{2H} - 2(j - i)^{2H} \right]. \tag{28}$$

Again, for sufficiently large $t$ and small $\delta \tau$, the auto-correlation function depends only on the difference between $s$ and $t$. Fig. 2 shows a comparison of $c(t, s)$ calculated with $\delta \tau = 0.05$ and $\delta \tau = 0.01$. Although the individual terms in Eq. (28) are dependent of $\delta \tau$, $c(t, s)$ remains unaffected.

Fig. 2. (a) Numerical evaluation of $c(t, s)$ for $H = 1/3$ using Eq. (28) with $\delta \tau = 0.05$. (b) Same as (a) with $\delta \tau = 0.01$. 

Apart from a constant factor, the structure function of the fractional OU process is determined by the choice of \( H \), since

\[
\langle dX_b dX_b \rangle = (k_b \omega_t \, dr)^2 - \langle 2k_b \omega_t \, dr \, d\Omega_b \rangle \\
+ (D_b d\Omega_b d\Omega_b) \\
= (dr^2 + (dr^{-H}) + D_b \, dr^{2H}) \\
- D_b \, dr^{2H}.
\]

(29)

Some formal progress can also be made by using the spectral representation of the fractional Langevin equation. For this purpose, we write Eq. (4) as

\[
\frac{dX_b}{dt} = -k_b \omega_t + \omega_b.
\]

(30)

although this is strictly speaking incorrect, since \( \Omega_b \) is nondifferentiable. The Fourier transform of (30) gives

\[
\chi_b(f) (2\pi E + k_b) - \sqrt{D_b} \omega_b(f),
\]

(31)

where \( \chi_b(f) \) and \( \omega_b(f) \) are the Fourier transforms of \( \chi_b(t) \) and \( \omega_b(t) \), respectively. It follows that the power spectrum of \( \chi_b \) is

\[
F_{\chi_b}(f) = \frac{D_b \omega_b(f)}{k_b + (2\pi f)^2}.
\]

(32)

The variance of \( \chi_b \) is given by

\[
\sigma^2_{\chi_b} = \int_{-\infty}^{\infty} F_{\chi_b}(f) \, df = \int_{-\infty}^{\infty} \frac{D_b \omega_b(f)}{k_b + (2\pi f)^2} \, df.
\]

(33)

Note that \( S_{\omega_b}(f) \) is the inverse Fourier transform of the auto-covariance function of fractional Gaussian noise. We have that

\[
\sigma^2_{\chi_b} = \frac{D_b}{2k_b} \int_{-\infty}^{\infty} c_{\omega_b}(\tau) e^{-k_b \tau} \, d\tau.
\]

(34)

where \( c_{\omega_b}(\tau) \) is the auto-covariance function of the fractional Gaussian noise. For \( H = 1/2 \), \( c_{\omega_b}(\tau) = \delta(\tau) \) and thus \( \sigma^2_{\chi_b} = D_b/2k_b \); this result is consistent with Eq. (26). For \( H > 1/2 \), the variance can be determined exactly in the limit of \( \lambda \rightarrow 0 \). In this case the auto-covariance function of fractional Gaussian noise is given by

\[
c_{\omega_b}(\tau) = H(2H - 1)\tau^{2H - 2}
\]

(35)

and from Eqs. (34) and (35) it is found that

\[
\sigma^2_{\chi_b} = H(2H - 1)\Gamma(2H - 1) D_b k_b^{-2H}.
\]

(36)

For \( H = 1/2 \), the situation is less obvious. The inverse Fourier transform of (32) leads to an interesting differential equation which relates the auto-covariance function of the fractional OU process, \( c_{\chi_b} \), to that of the fractional Gaussian noise, \( c_{\omega_b} \).

\[
k_b^2 c_{\chi_b}(\tau) - \frac{\partial^2 c_{\chi_b}(\tau)}{\partial \tau^2} = D_b c_{\omega_b}(t).
\]

(37)

It can be shown that if \( c_{\chi_b}(\tau) \) is an exponential function then \( c_{\chi_b}(\tau) \) is a \( \delta \)-function. This is the case of the conventional OU process. For other values of \( H \), the solution of (37) requires further investigations.

3.2. Numerical tests

We now investigate the statistical behaviour of the fractional OU process by numerically integrating Eq. (4). The numerical integration shows as previously indicated, that the fractional OU process is a stationary Gaussian process with a finite variance as time approaches infinity. \( H \) is chosen to ensure that the small-scale statistical behaviour of the fractional OU process is comparable with that of the physical process it simulates. The cases of \( H = 1/3, 1/2 \) and \( 2/3 \) are of particular interest because the structure functions given by Eq. (29) coincide respectively with those of Eulerian turbulence (12), Lagrangian turbulence (11) and the pressure fluctuations (in the Eulerian framework) for which the structure function in the inertial subrange is given by

\[
D_{pp}(\tau) = C_p \rho^2 (\bar{u} \tau)^{1/3}.
\]

(38)

where \( C_p \) is a universal constant and \( \rho \) is the density of fluid [13]. If further details are required, the choice of \( H \) may follow the results of [7].

For the numerical integration of Eq. (4), it is
important to ensure that the correlation properties of fractional Gaussian noises are accurately preserved by the noise generator. For $H < 1/2$, two noise generators are available based on the spectral technique described in [18] and the technique of successive random addition [19]. For $H > 1/2$ an additional noise generator, as suggested in [20], can also be used. Comparisons show that different noise generators may produce substantially different model outcomes. Intrinsic problems of the noise generators have been discussed by Fox [21].

Fig. 3 shows the time evolution of the probability density functions of the fractional OU processes for the cases $H = 1/3, 1/2$ and $2/3$. In these examples, we have chosen the simplest drift and diffusion coefficients with $k_n = 1$, $D_n = 1$ (for the determination of $k_n$ and $D_n$, see Section 5), and $dt = 0.01$. (A test with $dt = 0.001$ showed no difference compared with the results presented in Figs. 3 and 4.) A noise generator based on a spectral technique modified from that described in [18] was used for $H = 1/3$ and $1/2$, while for $H = 2/3$ the noise generator based on successive random addition was used, for reasons discussed later (see Fig. 5). For all three different $H$ values, the fractional OU process approaches a stationary Gaussian distribution with zero mean and a finite variance at sufficiently large time. For $H = 1/3$, the process approaches stationarity more rapidly than the $H = 1/2$ case, while for $H = 2/3$, the process approaches stationarity more slowly.

The first four statistical moments for the three cases are as presented in Fig. 4, which confirms that the fractional OU processes are indeed stationary Gaussian processes. The mean and the skewness are zero for all three cases, while the variance and the corresponding kurtosis obey the relationship

$$\langle \chi_n^4 \rangle = 3 \langle \chi_n^2 \rangle^2$$  (39)

as expected for a Gaussian process. The variance and kurtosis for $H = 1/3$ are smaller than those for $H = 1/2$, and for $H = 2/3$ larger. As indicated in Fig. 4 for $H = 1/3$ and $1/2$, there is excellent agreement between the model results and the predictions obtained from Eq. (26) (for $H = 2/3$ also Eq. (36), since smoothed and unsmoothed processes for $H > 1/2$ are not significantly different), while for $H = 2/3$ the model produces a variance and kurtosis somewhat smaller than the predictions. This disagreement is possibly due to the problems embedded in the fractional Gaussian noise generator. The other two noise generators mentioned previously have also been used for comparison, but neither produced satisfactory results.

The auto-correlation functions of the fractional OU processes are illustrated in Fig. 5, showing
Fig. 4. The first four moments of the fractional Ornstein–Uhlenbeck process for $H = 1/3$, $1/2$ and $2/3$. Variance and kurtosis for $t \to \infty$ estimated from Eq. (26) are indicated on the left hand size axes (dashed lines). The parameters are as for Fig. 3.

both numerical integration and Eq. (28). The auto-correlation function for the $H = 1/2$ case is an exponential function as expected. For $H \neq 1/2$, an expression for the auto-correlation function simpler than Eq. (28) is not available. For $H = 1/3$, the auto-correlation function calculated from numerical simulation and that estimated from Eq. (28) have excellent agreement. (The simulation cannot be distinguished from Eq. (28) in Fig. 5.) The results from both methods indi-

Fig. 5. Auto-correlation functions for $H = 1/3$, $1/2$ and $2/3$ obtained from numerical simulation and estimated from Eq. (28). For $H = 1/3$ and $1/2$, the spectral technique (S) is used for generating fractional Gaussian noise, while for $H = 2/3$ both the spectral technique and the successive random addition technique (RA) are used. The parameters are as for Fig. 3.
cate that the auto-correlation is positive for small $\tau$, then becomes negative for larger $\tau$ before approaching zero. The numerically simulated auto-correlation function for $H = 2/3$ agrees well with that estimated from Eq. (28) only for small $\tau$; they differ at large $\tau$. The three noise generators mentioned previously were applied to the model. The simulation based on the successive random addition technique produced a stronger correlation, while that based on the spectral technique produced a weaker correlation than expected. The simulation based on Mandelbrot's technique is numerically slow and produced an unrealistic oscillating auto-correlation function for large $\tau$ (not shown). These results show again that it is important to ensure an adequate fractional Gaussian noise generator, and that the current noise generators require further improvement before they can be used in Eq. (4) for cases with $H > 1/2$.

The energy spectra of the sample paths for the three cases are shown in Fig. 6. In the high frequency range, the spectral behaviour obeys the power law like $fF(X_f) \sim f^\alpha$. For $H = 1/3, 1/2$ and $2/3$, $\alpha$ is $-2/3, -1$ and $-4/3$, respectively. The spectra have a well-defined spectral peak and the spectral energy density decreases to the low frequencies. The spectral behaviour of the simulated sample path is similar to those often observed in the real atmosphere, corresponding to the inertial subrange of the Eulerian turbulence, Lagrangian turbulence and pressure fluctuations.

4. The choice of $D_b$ and $k_b$

For Ito-type Lagrangian stochastic models there is a well-established procedure for determination of the drift and diffusion coefficients [9]. The procedure involves Kolmogorov's theory of local isotropy and the solution of a Fokker–Planck equation. For Lagrangian turbulence, the diffusion coefficient $D$ is constrained by the requirement that for a time increment much larger than Kolmogorov's time scale, but much smaller than the Lagrangian time scale, the small-time behaviour of the simulation must obey Kolmogorov's hypothesis on the structure function in the inertial subrange. In the one-dimensional case, it follows that

$$D = C_0 \varepsilon. \quad (40)$$

The constraint on the drift coefficient involves manipulation of the Fokker–Planck equation. For homogeneous Gaussian turbulence in the Lagrangian framework, $k$ is equal to $D/2\sigma^2$.

Similarly, the two parameters in the fractional Langevin equation need to be specified so that the model can be used to reproduce the required statistical features of the physical processes. For the determination of the diffusion coefficient $D_b$ in Eq. (4), we require that the small-scale behaviour of turbulence obeys the theory of local isotropy as stressed throughout this study. Again, we consider the special Eulerian sample path along the mean wind direction. For the simulation of homogeneous Eulerian turbulence, the requirement on $D_b$ is that

$$D_b = C_1 (\varepsilon u)^{2/3} \quad (41)$$

if $u(t)$ is $u_1$, and

![Fig. 6. Energy spectra of the fractional Ornstein–Uhlenbeck process with the same parameters as for Fig. 3.](image)
The drift coefficient determined in this manner ensures the correct closure of the fractional OU process. The correctness of the other higher order moments is also then ensured because the fractional OU process is Gaussian.

Of course, for $H = 1/2$, $\gamma$ is 1 for all values of $k_n$. The dependence of $\gamma$ on $k_n$ turns out to be simple also for other values of $H$. Suppose that $\gamma$ can be expressed as $\gamma = c\gamma k_n^{1/2}$ with $c\gamma$ being a dimensionless constant; $\mu$ can then be determined on dimensional arguments. Since $k_n$ has the dimension $dt^{-1}$ and $k_n$ the dimension $dt^{-2H}$, the dimension of $\gamma$ is $dt^{-2H}$, and therefore

$$\mu = \frac{1 - 2H}{2H}.$$  \hspace{1cm} (45)

It is readily found from the above equation that $\mu$ is $1/2$ for $H = 1/3$, 0 for $H = 1/2$, and $-1/4$ for $H = 2/3$. Three examples of $\gamma(k_n, H)$ determined numerically with $H = 1/3$, 1/2 and 2/3, are shown in Fig. 7.

From Eq. (36), $k_n$ can be readily found for $H > 1/2$ in the limit of $\lambda \to 0$.

$$k_n'' = 2H(2H - 1)\lambda''(2H - 1)k_n.$$  \hspace{1cm} (43)

where $k_n = D_n/2\sigma^2$. It is useful in general to express $k_n$ as $k_n = \gamma k_n''$ with $\gamma$ being a dimensional parameter depending on $k_n$ and $H$. The value of $\gamma$ can be determined from Eq. (26) by numerical iteration.

The drift coefficient determined in this manner ensures the required variance of the fractional OU process. The correctness of the other higher order moments is also then ensured because the fractional OU process is Gaussian.

We return now to our main interest, the stochastic simulation of Eulerian turbulence. In Fig. 8, the statistics of the simulated Eulerian turbulence are shown along with those of the simulated Lagrangian turbulence. In these examples, more realistic parameters are used: the energy dissipation rate is $\varepsilon = 0.001414 \text{ m}^2 \text{s}^{-3}$, the variance $\sigma^2 = 0.5 \text{ m}^2 \text{s}^{-2}$ and the advection velocity $\bar{u} = 4\sigma$. (This can be considered as typical for atmospheric boundary layers; the average inverse turbulence intensity index $1/i = \bar{u}/\sigma$ for 18 runs considered by Hanna [22] is 3.76.) For Eulerian velocity fluctuations perpendicular to the mean wind we have $D_n = 0.05 \text{ m}^2 \text{s}^{-3}$, following Eq. (42) by choosing $C_1 = 1.5$ (c.g. [23]), $k_n = D_n/2\sigma^2 = 0.05 \text{ m}^2 \text{s}^{-3}$ and with the correction factor $\gamma = 0.19042$ estimated from Eq. (46). For Lagrangian turbulence with the same energy dissipation rate and velocity variance, we have $D_n = C_0 \varepsilon = 0.00707 \text{ m}^2 \text{s}^{-3}$ (by choosing $C_o = 5$, Hanna’s observation [22] shows that $C_0$ falls between 2 and 6), $k_n = D_n/2\sigma^2 = 0.00707 \text{ m}^2 \text{s}^{-3}$, and $\gamma = 1$ (dimensionless in this case). Fig. 8 shows that over a sufficiently large time the first four moments of the Eulerian and Lagrangian statistics are the same as required.

Fig. 9 compares the simulated Eulerian and
Fig. 7. Correction factor $\gamma$ as a function of $k_0$ for three $H$ values. The symbols are calculated and the curves fitted by using the least squares technique.

Fig. 8. First four moments of fractional Ornstein-Uhlenbeck process with $H = 1/3$ (Eulerian turbulence), $r = 0.01414 \text{ m}^2 \text{s}^{-1}$, $\sigma^2 = 0.5 \text{ m}^2 \text{s}^{-1}$, and the correction factor $\gamma = 0.19842 \text{ s}^{-1/4}$, compared with the corresponding case with $H = 1/2$ (Lagrangian turbulence). The spectral technique is used to generate fractional Gaussian noise.
Lagrangian auto-correlation functions. The Eulerian auto-correlation function for \( u_t \) is designated as \( R_{\text{II}} \) and that for \( u_n \) as \( R_{\text{nn}} \). Our knowledge of these functions is based either on theoretical models or on wind tunnel observations. In simple turbulent flows (isotropic, homogeneous and stationary), \( R_{\text{II}} \) is assumed to be positive with no zero-crossing, while \( R_{\text{nn}} \) has one zero-crossing [21]. Wind tunnel observations made by Tritton [25] support these conclusions. Fig. 9 shows that the Lagrangian auto-correlation function is again exponential, while the Eulerian auto-correlation functions cannot be exactly exponential. For small time lags, Eulerian fluctuations are positively correlated but for large time lags they are negatively correlated before becoming independent. As \( R_{\text{nn}} \) is concerned (\( R_{\text{nn}} \) case 1 in Fig. 9), this result is in agreement with the wind tunnel observations of Tritton [25] and the large eddy simulation of Sykes and Henn [26]. However, the model gives a nonequivocal zero-crossing auto-correlation function \( R_{\text{II}} \) which does not agree with the perception that \( R_{\text{II}}(r) \) remains positive for all \( r \). It is clear from Eqs. (41) and (42) that the only difference in modelling \( u_t \) and \( u_n \) lies in the fact that \( D_n \) differs by a factor of \( 4/3 \), and such a quantitative difference does not result in a qualitative difference in \( R_{\text{II}} \) and \( R_{\text{nn}} \).

Nevertheless, the model shows that \( u_t \) has a correlation time scale about 1.6 times longer than that of \( u_n \).

We now consider the integral time scale of the fractional OU process. The energy spectrum of continuous fractional Gaussian noise is given by Eq. (9), and from (32) the integral time scale of the fractional OU process is given by

\[
T_n = \frac{F_{\text{in}}(0)}{2\sigma_{u_n}^2} = \begin{cases} 
0 & H < 1/2, \\
\frac{D_n}{2\sigma_{u_n}^2 k_{h_n}^2} & H = 1/2, \\
\propto & H > 1/2.
\end{cases} \tag{47}
\]

For discrete fractional Gaussian noise, Mandelbrot and Van Ness [4] showed that \( F_{\text{in}}(0) \) does not obey Eq. (9) as \( f \) approaches \( 0 \) and in particular

\[
F_{\text{in}}(0) = \begin{cases} 
\frac{2\lambda^{2H-1}}{2H + 1} & H \leq 1/2, \\
\propto & H > 1/2.
\end{cases} \tag{48}
\]

As a result, the integral time scale of the fractional OU process is given by

\[
T_n = \frac{F_{\text{in}}(0)}{2\sigma_{u_n}^2} = \begin{cases} 
\frac{D_n \lambda^{2H-1}}{\sigma_{u_n}^2 k_{h_n}^2(2H + 1)} & H \leq 1/2, \\
\propto & H > 1/2.
\end{cases} \tag{49}
\]
The results given in Eqs. (47) and (49) are consistent for $H \geq 1/2$. The $H > 1/2$ case is better known in hydrology (for instance, the Hurst process). However, for $H < 1/2$, Eq. (49) indicates that $T_h$ is finite and depends on $\lambda$. In particular, the result of $T_h \rightarrow \infty$ for $\lambda \rightarrow 0$ given in Eq. (51) contradicts the result of $T_h = 0$ for $\lambda = 0$ given in Eq. (47).

The mathematical subtlety associated with fractional Gaussian noise for $H < 1/2$ reflects the unrealistic physical meaning of the noise (which remains correlated as $t \rightarrow \infty$) in modelling a physical process. However, the fractional OU process is still an adequate representation of Eulerian turbulence. In case of the conventional OU process, for which the auto-correlation function decays exponentially with time, $T_h$ can be interpreted as the time scale within which the signals are significantly correlated. However, such an interpretation may not be entirely adequate for $H \neq 1/2$, because $T_h$ depends on how fast the signals become uncorrelated as time increases and thus reflects how signals are correlated for large times. Therefore, it is sensible not to choose the integral time scale but the time required for the auto-correlation function to be sufficiently small (say 0.37) as the correlation time scale. Such a definition is often used in practice because the length of experimental measurements is always limited. As far as the fractional OU process is concerned, the correlation time scale defined in this way is independent of $\sigma$, as already shown in Figs. 1 and 2.

If we ignore the weak negative correlation at large time delays, and define the Eulerian time scale as the time required for the auto-correlation function to be zero, the Eulerian time scale $T_E$ (from $R_{nn}$ case 1) is 55 s and the Lagrangian time scale $T_L$ is 141 s. Thus as expected, the Eulerian turbulence has a shorter correlation time scale than the Lagrangian turbulence and the ratio between $T_L$ and $T_E$ is 2.6. A considerable number of observations in the atmosphere have been devoted to determining the value of $T_L / T_E$ (e.g. [22]). The estimated values of this ratio vary considerably and may depend on the stability (turbulence intensity) of the atmosphere. Based on observational evidence, Hanna [22] suggested that

$$\frac{T_L}{T_E} = 0.7 \frac{\bar{u}}{\sigma}. \quad (50)$$

Using Hanna’s empirical finding, the ratio $T_L / T_E$ for the present case is 2.8, which agrees remarkably well with the model result of 2.6. If $\bar{u}$ is assumed to be 10 $\sigma$, then $T_E$ (from $R_{nn}$ case 2) would be around 22 s (Fig. 9) and the ratio $T_L / T_E$ is 6.4, again agreeing remarkably well with Hanna’s prediction of 7. These comparisons show that Eq. (4), with the parameters chosen as described above, not only ensures that the Lagrangian and Eulerian moments are identical and produced acceptable Lagrangian and Eulerian auto-correlation functions, it also maintains a ratio between the Lagrangian and Eulerian time scales which agrees with the observations.

5. Concluding remarks

In this paper, a new type of stochastic differential equation (fractional Langevin equation) with fractional Gaussian noise has been proposed for the simulation of homogeneous atmospheric turbulence. For the particular choice of the exponent parameter $H = 1/2$, the fractional Langevin equation is reduced to the conventional Langevin equation (an Ornstein-Uhlenbeck process in one dimension) which is known to be adequate for the simulation of homogeneous Lagrangian turbulent velocities. However, the major advantage of the fractional Langevin equation lies in its ability to model turbulent fluctuations with more peculiar characteristics. Of particular value is the ability of the fractional Langevin equation to simulate the Eulerian turbulence field, with $H = 1/3$. Most of the statistical properties of Eulerian turbulence known from observations can be correctly reproduced, including the intensity of fluctuation...
(variance), the Gaussian probability distribution, the auto-correlation function, and spectral density function (−2/3 slope in the inertial subrange). The successful simulation of Eulerian turbulence is important for Lagrangian stochastic models of heavy particle diffusion for which an adequate description of the trajectory-closing effect is essential [3].

The fractional Langevin equation is a practical step forward and is an improvement on attempts to directly apply fractional Brownian motions to the stochastic simulation of Eulerian turbulence as in [5]. Such an equation preserves the capability of modelling turbulence with fractional spectral slopes (−2/3, −4/3, etc.) in the inertial subrange, while enabling the stochastic process to be stationary with finite energy. In a more general sense, this paper shows an example of applying fractional Brownian motions to a process which has self-similar properties limited to a range of scales instead of all scales. It is more often the case in nature that the fractal dimensions of the process vary with the time or length scale.

Although it is possible to generate statistically stationary random processes with known spectral behaviour by summing random Fourier modes as in Tung et al. [27], the fractional Langevin equation provides a simpler alternative. It is suggested that for homogeneous turbulence, the diffusion coefficient of the fractional Langevin equation is constrained by Kolmogorov's theory of local isotropy while the drift coefficient is determined by requiring stationarity and the statistical moments. Numerical tests showed that these constraints on the diffusion and drift coefficients are both simple and efficient. Thus, a stochastic representation of homogeneous Eulerian velocity turbulence requires only the energy dissipation rate and the variance as specified parameters. The simplicity of the model is a desirable feature if such a scheme is to be incorporated in more complicated models such as those for heavy particle diffusion.

In this study, analysis and tests are provided for a simple case of fractional stochastic differential equations. It is obviously desirable to extend Eq. (4) to a more general form for studying more complex turbulence, especially non-Gaussian turbulence. For non-Gaussian turbulence, it may be necessary to extend the fractional Langevin equation to more general forms with nonlinear drift coefficients, which is then directly comparable with the Ito-type stochastic differential equation. However, the mathematical tools available for Ito-type stochastic models are not necessarily directly applicable to fractional stochastic differential equations. The major mathematical difficulty lies in the fact that fractional Gaussian noises are correlated, and the stochastic processes determined by Eq. (4) are not Markovian. Thus, to extend the fractional Langevin equation for the representation of non-Gaussian turbulence require further investigations. Despite the fact that the fractional stochastic differential equations are complex and mathematically less tractable than Ito-type stochastic models, they can be useful models in practice. This study represents a preliminary step in this direction.

Acknowledgments

I wish to thank Dr. M.R. Raupach for his encouragement and valuable advice throughout this study. I am grateful to Drs. M. Westcott, M.S. Borgas and B.L. Sawford for helpful discussions.

References
